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PRESSURE EFFECTS ON SEPARATION IN AXISYMMETRIC STOKES FLOWS

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Abstract—Two Stokes flows which are known to lead to separation are reconsidered from a more dynamic perspective, and it is found that within the region of separated flow there is an extremum for the pressure. A simple argument is presented which indicates that this is true under reasonable conditions.

1. INTRODUCTION

One of the interesting developments in the study of slow viscous flows, with consequences for all fluid mechanics, has been the recent examination of separation. Much of this work of the past decade has been reviewed in the articles of O'Neill & Ranger (1979), and Hasimoto & Sano (1980). In the early development of Stokes flows much of the interest centred on the drag and, where relevant, the torque produced by a particular shaped body in either a uniform or simple shear flow. Many sophisticated analytical techniques have been developed for these calculations and a good understanding of the role of the body shape and dimensions in influencing the drag has been gained.

The existence of separated flows, when the Reynolds number is effectively zero, was first shown by Dean (1944), though only in the last few years has a catalogue of different separated flows been developed to give a clearer indication of the situations under which separation can be anticipated. These studies have been through a more detailed examination of the stream function. Through a combination of analytical and numerical methods, invariably very lengthy, stagnation and separation points in the fluid have been located, and an overall plotting of the streamlines in the vicinity of the body sketched. General results have not come easily though. Interest has tended to concentrate on the dividing streamline, where this intersects the body at the point of separation for example, in what can be considered as essentially a study of the kinematic aspects of the flow. This is natural given the historical perspective of high Reynolds number flows past bluff bodies, where local considerations in a boundary layer context were just about all that were possible. However, it can be argued that the precise position of the point of separation is of less consequence in a low as against a high Reynolds number flow. It is known that the flow solution to the Stokes equation is analytic at the point of separation, and there are not the same immediate consequences for the values of the body drag once separated flow commences; there is a smooth transition through the states at which separation develops, and the expression for the drag does not appear to notice the precise onset of separation.

The purpose of the present paper is to re-examine some of the problems recently considered in the literature from a more dynamic perspective. Specifically, we evaluate the pressure for the uniform flow past two different axisymmetric geometries to note any effects which follow from the onset of separation. These are the flow past two spheres (Davis *et al.* 1976), and the flow past a spherical cap (Dorrepaal *et al.* 1976). The major observation which can be made from these calculations is that within the region of separated flow there is an extremum for the pressure function. (Because of the reversibility of all Stokes flows this can be either a maximum or a minimum.) It was seen from the earlier discussions that in these situations the fluid does not continue to follow the contours of the body, but rather separates at some point on the surface. It would appear that there is insufficient momentum to complete the flow around the body, and the existence of the pressure extremum on the axis is thereby connected to the separation phenomenon. For Stokes flows it is well known that the pressure is a harmonic function, and potential theory states that the extremum for the pressure must lie on the "boundary", which implies either the surface of the body or the axis of symmetry in axisymmetric situations. If separation is not present the extrema must be on the body. In the first problem, when separation only occurs as the distance between the spheres diminishes, it can be shown that the onset of separation is marked by the occurrence of the pressure extremum at the stagnation point on the body along the axis.

It would be attractive if a general argument could be developed which shows that there is always an extremum for the pressure when the flow past the body leads to separation. However, this has not proved to be possible. A restricted argument, with the requirement that the separation bubble satisfies a concavity condition, has been sketched in the last section, using a local analysis in the neighbourhood of the stagnation points. A general argument must be global in character, and this has not been attained.

No general results of a similar nature occur for all two-dimensional flows. For example, in the case of a shear flow past a circular cylinder (see Smith 1979) it can be seen that in some cases, depending on the flow parameters, separation does lead to the existence of an extremum for the pressure, in others it does not.

In a final introductory comment, Ranger (1971) showed that separation can take place for the axisymmetric flow past a rotating sphere when the secondary flow due to the rotation and the streaming velocities have equal orders of magnitude. In this situation the pressure for the secondary flow is not harmonic, and it can be shown that for a certain range of the flow constants the pressure has an extremum away from the axis of symmetry, and outside the domain of separated flow. However, it has been shown that a pressure extremum does exist within the separated region, though the details are not presented here.

If $a\rho$, az represent radial and axial distances in a cylindrical co-ordinate system, where a is the length scale, then the Stokes equations are

$$Rp_\rho = \zeta_z, \qquad -Rp_z = \rho^{-1}(\rho\zeta)_\rho \tag{1.1}$$

where p is the pressure, ζ the vorticity (both non-dimensional) and R is the Reynolds number. When ψ is the stream function, then $\zeta = -\rho^{-1}L_{-1}(\psi)$, where $L_{-1} \equiv (\partial^2/\partial\rho^2) - (1/\rho)(\partial/\partial\rho) + (\partial^2/\partial z^2)$. On eliminating the pressure, there follows the basic fourth order equation $L^2_{-1}(\psi) = 0$; this is the equation which is solved along with the no slip boundary conditions on the surface of the body to give an understanding of the streamlines, and to calculate the drag.

2. PARTICULAR CASES

(a) *Two spheres.* Here we follow the work of Davis *et al.* (1976). The spheres of radius a have centres on the axis of symmetry at $z = \pm d$; the uniform stream with velocity U is parallel to the z-axis. The bispherical co-ordinate system

$$\rho = \frac{c \sin\eta}{\cosh\xi - \cos\eta}, \qquad z = \frac{c \sinh\xi}{\cosh\xi - \cos\eta} \tag{2.1}$$

is introduced so that the spheres are represented by $\xi = \pm\alpha$, where $a = c \operatorname{cosech}\alpha$, $d = c \coth\alpha$. Then Davis *et al.* showed that

$$\psi = \frac{c^2 \sin^2\eta}{2(\cosh\xi - \cos\eta)^2}$$

$$+(\cosh\xi-\cos\eta)^{-3/2}\sum_{n=1}^{\infty}\left\{A_n\cosh\left(n-\frac{1}{2}\right)\xi+C_n\cosh\left(n+\frac{3}{2}\right)\xi\right\}$$

$$\times\{P_{n-1}(\cos\eta)-P_{n+1}(\cos\eta)\} \qquad [2.2]$$

where

$$A_n=-\frac{n(n+1)}{\sqrt{(2)}(2n-1)(2n+1)}\left\{\frac{2(1-e^{-(2n+1)\alpha})+(2n+1)(e^{2\alpha}-1)}{2\sinh(2n+1)\alpha+(2n+1)\sinh 2\alpha}\right\}, \qquad [2.3]$$

$$C_n=\frac{n(n+1)}{\sqrt{(2)}(2n+1)(2n+3)}\left\{\frac{2(1-e^{-(2n+1)\alpha})+(2n+1)(e^{2\alpha}-1)}{2\sinh(2n+1)\alpha+(2n+1)\sinh 2\alpha}\right\}. \qquad [2.4]$$

Now long, but straightforward calculations show that

$$\rho\zeta=c^2(\cosh\xi-\cos\eta)^{-1/2}\sum_{n=1}^{\infty}D_n\cosh\left(n+\frac{1}{2}\right)\xi\cdot\{P_{n-1}(\cos\eta)-P_{n+1}(\cos\eta)\}, \qquad [2.5]$$

with

$$D_n=\frac{2n(2n+3)}{2n+1}A_{n+1}-(2n-1)A_n+2(n+1)C_n-\frac{2(n+1)(2n-1)}{2n+1}C_{n-1} \qquad [2.6]$$

for $n=1,2,3,\ldots$; $C_0=0$. Solving [1.1] for the pressure shows finally that

$$Rp=c^2(\cosh\xi-\cos\eta)^{1/2}\sum_{n=0}^{\infty}E_n\sinh\left(n+\frac{1}{2}\right)\xi\cdot P_n(\cos\eta), \qquad [2.7]$$

where E_n is given in terms of D_n by the difference equation

$$n(E_n-E_{n-1})=(2n+1)D_n-(2n-1)D_{n-1} \qquad [2.8]$$

for $n=1,2,3,\ldots$; $D_0=0$. When we require E_n to tend to zero exponentially as $n\to\infty$, a unique solution is obtained for E_n from (2.8) given by

$$E_n=\frac{2n+1}{n+1}D_n-\sum_{k=n+1}^{\infty}\frac{2k+1}{k(k+1)}D_k,\quad n=0,1,2,\ldots. \qquad [2.9]$$

It has not been found possible to simplify [2.9] using [2.3, 4, 6], and so numerical values for particular α must suffice at this point. Also, the Watson transformation, which was of crucial help to Davis *et al.* in improving the numerical accuracy of their calculations, cannot readily be utilised. Nevertheless, sufficient accuracy could be attained by direct calculations to firmly establish the position of the pressure extremum within the separated region.

The particular value $\alpha=1$ has been taken; according to Davis *et al.*, separation takes place for this value, and the separated region joins both spheres. The calculations show $E_0\simeq-0.367650$, $E_1\simeq0.184242$, $E_2\simeq0.144709$, $E_3\simeq0.032130$, $E_4\simeq0.005487$, $E_5\simeq0.000907$, $E_6\simeq0.000147$, $E_7\simeq0.000024$, $E_8\simeq0.000004$, $E_9\simeq0.000001$; these show a sufficiently rapid decay for satisfactory numerical accuracy. The lines $p=$ constant are sketched in figure 1 in the half-space $z\geq0$ only; there is a fore–aft symmetry about $z=0$. The flow is considered to be vertically down and the dotted line indicates the boundary

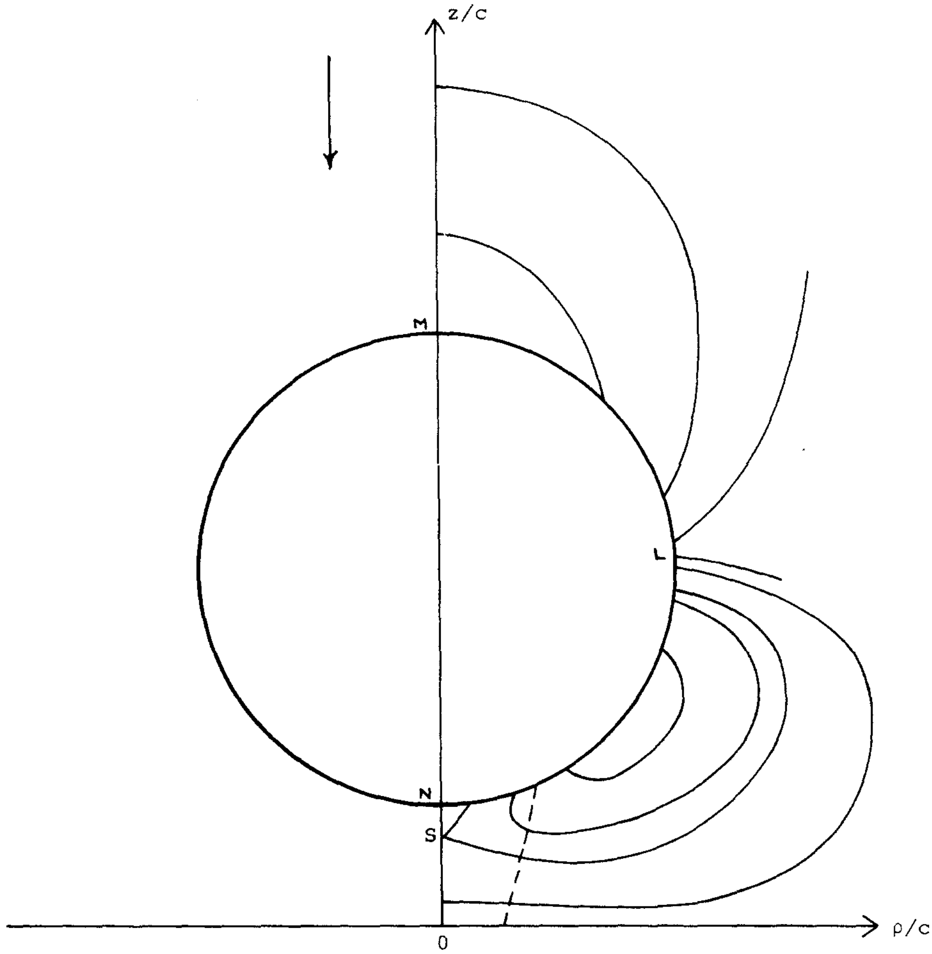

Figure 1.

of the separated region. Within the upper half plane $z \geq 0$ the pressure has an absolute maximum at the forward stagnation point M, and an absolute minimum at the point N. The base pressure has been adjusted so that p is zero along $z = 0$, and the curve $p = 0$ also meets the sphere at L. The local extremum for $p(\rho, z)$ is at the point S within the separated region; S is a saddle point for the function p. The streamline which borders the separated flow is also shown. The local extremum for the pressure can be noted within the separated region at the point $\rho = 0$, $z \simeq 0.36c$.

(b) *Spherical cap*. In this work we follow the results of Dorrepaal, O'Neill & Ranger (1976), which have recently been verified experimentally by Collins (1979). The cap occupies the domain $r = 1$, $0 \leq \theta \leq \alpha$ where r and θ are spherical polar coordinates with $z = r \sin\theta$, $\rho = r \cos\theta$. The equations corresponding to [1.1] in this coordinate system are

$$Rp_r = (r^2 \sin\theta)^{-1}\zeta_\theta, \qquad -Rp_\theta = (\sin\theta)^{-1}\zeta_r,$$

where

$$L_{-1} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}.$$

Now Dorrepaal *et al.* showed for $r < 1$ that

$$\psi = \frac{1}{2}r^2\sin^2\theta - \sum_{n=1}^{\infty} A_n^{(1)} r^{n+1}\{P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta\}$$

$$+ (r^2 - 1)\sum_{n=1}^{\infty}\left\{\frac{1}{2}(n+1)A_n^{(1)} - A_n^{(2)}\right\} r^{n+1}\{P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\}, \qquad [2.10]$$

where

$$A_n^{(1)} = \frac{1}{3\pi}\left\{\frac{\sin(n-1)\alpha}{n-1} - \frac{\sin(n+2)\alpha}{n+2}\right\} + \frac{\sqrt{(2)}C}{2n+1}\cos\left(n+\frac{1}{2}\right)\alpha$$

$$+ \frac{\sqrt{(2)}D}{2}\left\{\frac{\cos\left(n+\frac{3}{2}\right)\alpha}{2n+3} + \frac{\cos\left(n-\frac{1}{2}\right)\alpha}{2n-1}\right\},$$

$$A_n^{(2)} = \frac{1}{3\pi}\left\{\frac{\sin(n-1)\alpha}{n-1} - \frac{\sin(n+2)\alpha}{n+2}\right\} + \frac{\sqrt{(2)}B}{2n+1}\cos\left(n+\frac{1}{2}\right)\alpha$$

$$+ \frac{3\sqrt{(2)}D}{8}\left\{\frac{\cos\left(n+\frac{3}{2}\right)\alpha}{2n+3} + \frac{\cos\left(n-\frac{1}{2}\right)\alpha}{2n-1}\right\}$$

with $B = (\sqrt{(2)}s/24\pi)(9 + 2s^2)$, $C = (\sqrt{(2)}s/6\pi)(3 + 2s^2)$, $D = (3\sqrt{2}/2\pi)s$, when $s = \sin(\alpha/2)$; a similar formula follows for $r > 1$. Working through the calculations as before now shows

$$Rp = 2\sum_{n=1}^{\infty}\frac{(2n+1)(2n+3)}{n}\left\{\frac{1}{2}(n+1)A_n^{(1)} - A_n^{(2)}\right\} r^n P_n(\cos\theta) \qquad [2.11]$$

for $r < 1$ to complete the solution in this domain. When the equivalent expression is gained for $r > 1$, care must be taken in adding the correct arbitrary constant to ensure equality along $r = 1$ for $\alpha \le \theta \le \pi$.

We now take the particular value $\alpha = (1/2)\pi$ (the hemispherical cap), and the coefficients here can be simplified for [2.11] to give

$$Rp = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n r^{2n} P_{2n}(\cos\theta) + \frac{6}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(2n+1)(2n+3)} r^{2n+1} P_{2n+1}(\cos\theta)$$

for $r < 1$. This can be summed using the techniques outlined by Dorrepaal *et al.* The level curves for p in the azimuthal plane are sketched in figure 2, together with the streamline $\psi = 0$. The flow is taken to be from right to left, and the dotted line represents the separation streamline $\psi = 0$; 0 is the centre of the hemisphere. There is an absolute maximum for the pressure at M. Because of the sharp edge, there is also a square root singularity with $Rp \propto (r-1)^{-1/2}$ close to the rim of the cap, which leads to a concentration of equi-pressure lines near there. With the pressure at infinity taken to be zero, the line were $p = 0$ also intersects the hemisphere at L. Again, the presence of the extremum, as indicated by S at the point $r \simeq 0.67$, $\theta = 0$, is seen to be within the separated region, and the fact that S is a saddle point is confirmed.

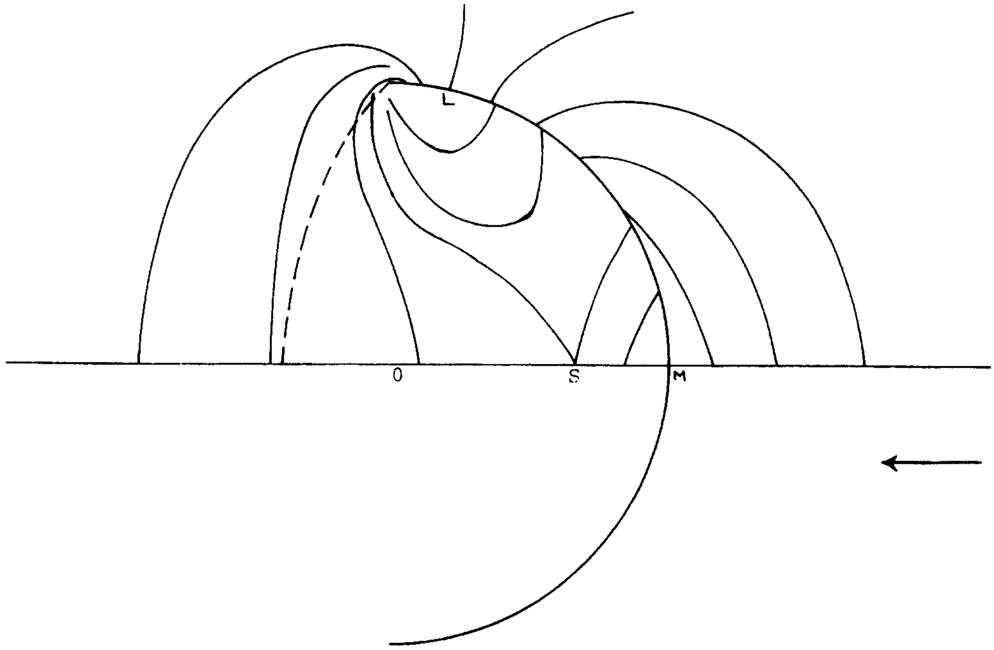

Figure 2.

3. GENERAL DESCRIPTION

For the general situation we consider the flow past a bluff body with a sufficiently smooth geometry close to the rear stagnation point, as represented in figure 3. The governing equations are written in the form

$$Rp_\rho = u_{\rho\rho} + \rho^{-1}u_\rho - \rho^{-2}u + u_{zz},$$

$$Rp_z = w_{\rho\rho} + \rho^{-1}w_\rho + w_{zz}, \qquad [3.1]$$

$$u_\rho + \rho^{-1}u + w_z = 0;$$

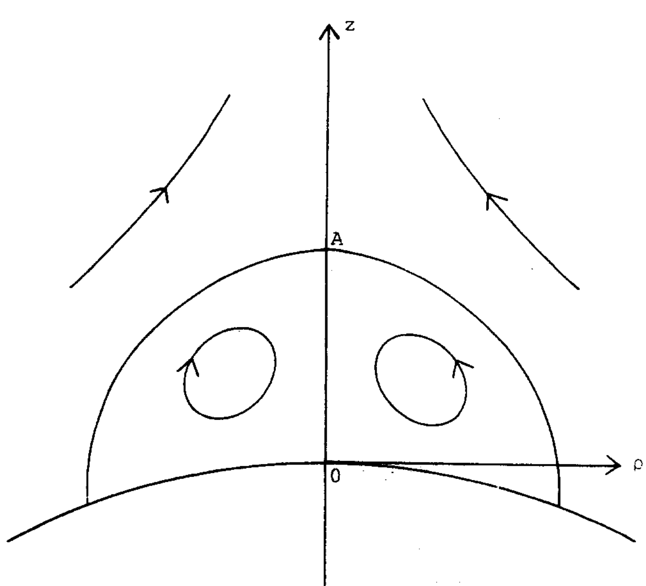

Figure 3.

u and v represent the radial and axial velocities. The axisymmetric nature of the flow necessarily leads to $p_\rho = 0$ for all points on the axis where $\rho = 0$.

At the rear stagnation point 0, the equation of the surface can be expressed in the form $z = -\alpha\rho^2 + 0(\rho^4)$ for some constant α, and close to 0 we can write

$$\psi = \mathscr{A}\rho^2\{z^2 + \alpha z\rho^2 + \mathscr{B}z^3 + \text{(higher order terms)}\} \qquad [3.2]$$

where $\mathscr{A}$ and $\mathscr{B}$ are constants; this representation ensures that ψ is zero on the surface to $0(\rho^4)$, and because there is an inflow towards 0, $\mathscr{A} < 0$. Substitution of [3.2] into [3.1] shows that Rp_z equals $4\mathscr{A}$ at 0, which is negative. The sign of α indicates the concavity of the surface through 0, and so the sign of Rp_z is independent of the shape of the body.

Further, A is also a stagnation point, and the free streamline $\psi = 0$ passes through A, so that we can write

$$\psi = \mathscr{C}\rho^2\{z_1 + \beta\rho^2 + \mathscr{D}z_1^2 + \text{(higher order terms)}\} \qquad [3.3]$$

close to A where $z_1 = z - c$ with $OA = c$; $\mathscr{C}$ is a positive constant. When (3.3) is substituted into (3.1) it follows that Rp_z equals $4\mathscr{C}(4\beta + \mathscr{D})$ at A, which is certainly positive for β, $\mathscr{D} > 0$. Now the restriction $\beta > 0$ is certainly satisfied when the dividing streamline is concave at A as shown in figure 3. To understand the role of $\mathscr{D}$, it is seen that, if we can approximate $\psi = 0$ near A by an ellipse in the ρ, z-plane with semi-axes of length a, b respectively, then we can utilise the fact that the stream function

$$\psi \propto \mathscr{C}\rho^2\left\{z_1 + \frac{z_1^2}{2b} + \frac{b\rho^2}{2a^2}\right\}$$

exactly represents the rotational flow within the spheroid to justify taking $\mathscr{D} > 0$.

In practise, this is a reasonable condition, and is satisfied in the particular problems considered; nevertheless, it is most likely extraneous, and would be removed if a more global argument were possible.

Also, we see that $p_{\rho\rho}p_{zz} - p_{\rho z}^2$ is equal to $2p_2p_0''$ on $\rho = 0$ when $p = p_0(z) + \rho^2 p_2(z) + 0(\rho^4)$. Because $p(\rho, z)$ is harmonic it follows that $4p_2 + p_0'' = 0$; therefore

$$(p_{\rho\rho}p_{zz} - p_{\rho z}^2)_{\rho=0} = -\frac{1}{2}p_0''^2 < 0,$$

and the extremum is always a saddle-point.

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